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The Signorini problem with Coulomb friction for a hyperelastic body under finite deformations☆

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ABSTRACT

The quasi-static three-dimensional problem of elasticity theory for a hyperelastic body under finite deformations, loading by bulk and surface forces, partial fastening and unilateral contact with a rigid punch and in the presence of time-dependent anisotropic Coulomb friction is considered. The equivalent variational formulation contains a quasi-variational inequality. After time discretization and application of the iteration method, the problem arising with "specified" friction is reduced to a non-convex miniumum functional problem, which is studied by Ball's scheme. The operator in contact stress space is determined. It is shown that a threshold level of the coefficient of friction corresponds to each level of loading, below which there is at least one fixed point of the operator. If the solution at a certain instant of time is known, the iteration process converges to the solution of the problem at the next, fairly close instant of time.

Contact and friction of the crack sides, in the problem of the growth of delamination,[1] has a considerable influence on the nature of the growth of a crack. The problem considered here, which corresponds to the case of thin-film separation, is therefore an urgent one.

There are numerous papers in which the solvability of the contact problem with friction has been investigated. However, in a large proportion of these, a regularized friction law rather than Coulomb's law was used, so that the list of key studies is short. The linear unilateral static problem of elasticity theory with friction, the equivalent variational inequality and the corresponding fixed-point problem were formulated,[2] and the solvability of the problem for a strip was proved using Tikhonov's fixed-point theorem.[3] For Coulomb's law, written in terms of the velocities, a converging iteration process has been proposed[4] for solving the variational inequality. It was suggested[5] that the normal contact stress should be regarded as a distribution, and that the fixed-point principle should be used. As a result of improvement of the procedure developed earlier,[3] the existence of a weak solution of the three-dimensional problem was proved.[6,7] A slightly different technique for obtaining a proof was then developed.[8] In a geometrically non-linear quasi-static contact problem with Coulomb friction, it was suggested that the iteration method should be used to solve the quasi-variational inequality.[9]

Note that the solvability of the static contact problem with Coulomb friction was proved[3–8] within the framework of linear elasticity theory, and then a much more complex problem (from the viewpoint of elasticity theory) was formulated,[9] without proof of solvability.

In the present paper, an attempt is made to investigate the solvability of the problem,[9] taking into account the anisotropy of friction. Using the iteration method,[9] it proved possible to change to non-convex minimization of the functional with specified friction. Difficulties associated with the non-convexity of the function of the specific strain energy were overcome by using the concept of the polyconvexity of the energy and corresponding mathematical results,[10,11] which enabled the complete continuity of the iteration operator for contact stresses to be proved. Below, estimates of the change in stresses at an iteration step are obtained, and the applicability of Tikhonov's fixed-point theorem is substantiated. The novelty of these estimates lies, for certainly, in their specific nature, but the convergence of the iteration method, the solvability of the problem and the possibility of using this method for calculations are nevertheless proved as a result.

1. Preliminary data and notation

The necessary data are given in well-known treatises.[10,12] Summation over dummy indices, direct tensor notation and the products $\mathbf{A}\cdot\mathbf{B}$, $\mathbf{A}\cdot\cdot\,\mathbf{B}$ and $\mathbf{A}\otimes\mathbf{B}$ (the dot, double-dot, and diad products) are used. Below, R^n is an n-dimensional Euclidean space with orthonormalized basis $\{\boldsymbol{i}_k\}_1^n$, $R_+ = (0, \infty]$, and $\int f(x)d\omega$ is the Lebesgue integral over the set ω.

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Before deformation, the body occupies in R^3 the region Ω, a bounded open connected Lipschitz set with boundary $\Gamma = \partial\Omega$. The vectors $\mathbf{r} = x^k\mathbf{i}_k$ and $R = X^k\mathbf{i}_k$ specify the position of a point mass before and after deformation. The function $\mathbf{R}(x)$ with a positive Jacobian describes the deformation, while the vector $\mathbf{u} = \mathbf{R} - \mathbf{r}$ is the displacement; $\mathbf{R}_\upsilon = \mathbf{r} + \upsilon$.

On the oriented area $\mathbf{n}d\sum$ with normal $\mathbf{N}$ after deformation for any vector $\mathbf{b}$, the folowing expansion holds

$$\mathbf{b} = b_N\mathbf{N} + \mathbf{b}_T, \quad b_N = \mathbf{N}\cdot\mathbf{b}, \quad \mathbf{b}_T = \mathbf{b} - b_N\mathbf{N}$$

For the stress on it, $\mathbf{t}_n = \mathbf{n}\cdot\mathbf{T}$, where $\mathbf{T}$ is the first Piola–Kirchhoff tensor, the formula $\mathbf{t}_n = t_N\mathbf{N} + \mathbf{t}_T$ holds.

The following notation is used below

$$\nabla = \{\mathbf{i}^k(\partial/\partial x^k)\}, \quad \operatorname{div}\mathbf{A} = \nabla\cdot\mathbf{A}, \quad \mathbf{DA} \equiv \operatorname{grad}\mathbf{A} = \nabla\otimes\mathbf{A}$$

Furthermore, when indicating a specific value of the vector $\mathbf{a}$, use is made of the functions $\mathbf{l}(\mathbf{a}) = \mathbf{a}/|\mathbf{a}|$ and $\mathbf{f}(\mathbf{a}) = -(\mathbf{F}\cdot\mathbf{l})$, where $\mathbf{F}$ is a positive-definite friction tensor, and also $f(\mathbf{a}) = |\mathbf{f}|$, $\mathbf{e}(\mathbf{a}) = \mathbf{f}/|\mathbf{f}|$ and $t_\mathbf{f}(\mathbf{a}) = ft_N$, and $\upsilon_\mathbf{f} = (\upsilon\cdot\mathbf{e})\mathbf{e}$ is the component of the vector υ in the direction $\mathbf{f}$.

For a hyperelastic body, $\mathbf{T} = \text{Э}_\xi(x, \mathbf{DR})$, where Э_ξ is a derivative with respect to the tensor $\mathbf{DR}$ of the specific strain energy $\text{Э}\,(x, \mathbf{DR})$.

2. Formulation of the problem

At the instant of time t, the boundary $\Gamma = \cup_k\Gamma_k^t$, where Γ_0^t is independent of t, mes $\Gamma_0^t > 0$, $\Gamma_k^t \cap \Gamma_s^t = \emptyset$, $k \neq s$ and $k, s = 0, 1, \ldots, 4$. The superscript t is omitted where possible. If the deformation $\mathbf{R}$ is fairly smooth, then the unit vectors of the outward normal $\mathbf{n}$ (Ref. 13, p. 88) to $\mathbf{\Omega}$ and $\mathbf{N}$ to $\mathbf{R}(\mathbf{\Omega})$ are defined almost everywhere on Γ.

We will give the equation of motion and the equation of state[12]

$$\nabla\cdot\mathbf{T} + \rho\mathbf{S} = \rho\ddot{\mathbf{u}} \text{ in } \Omega, \quad \mathbf{T} = \text{Э}_\xi(x, \mathbf{D}(\mathbf{r} + \mathbf{u})) \text{ in } \Omega \tag{2.1}$$

The dot denotes the total time derivative, and ρ is the mass density.

The simplest configuration of the boundary conditions is as follows:

$$\mathbf{u} = 0 \text{ on } \Gamma_0, \quad \mathbf{t}_n = \mathbf{P}_n \text{ on } \Gamma_1 \tag{2.2}$$

and $\mathbf{S}$ and $\mathbf{P}_n$ are the conservative mass and surface forces.

Unilateral contact (non-penetration) of the body and a rigid punch with the boundary $\Gamma_\psi = \{x|\psi(\mathbf{r}) = 0\}$, where there are the fairly smooth functions $\psi(\mathbf{r}) > 0$ outside and $\psi(\mathbf{r}) < 0$ inside the punch, occurs on a section of the boundary

$$\Gamma_2^t = \{x|\psi(\mathbf{R}(x,t)) = 0\}$$

Thus, on Γ_2^t, the following conditions must be satisfied

$$\psi(\mathbf{r}(x) + \mathbf{u}(x,t)) \geq 0 \text{ and } t_N \leq 0 \tag{2.3}$$

On the set $\Gamma_5^t = \{x|\, t_N < 0,\ t_T \neq 0\} \subset \Gamma_2^t$ there is contact with friction, and, according to the Amonton–Coulomb model of anisotropic friction,[14]

$$|\mathbf{t}_T| < |t_\mathbf{f}| \text{ и } \dot{\mathbf{u}}_T = 0 \text{ or } \mathbf{t}_T = |t_N|\mathbf{f} = |t_\mathbf{f}|\mathbf{e} \text{ и } \mathbf{l} = -\mathbf{F}^{-1}\cdot\mathbf{t}_T/|t_N| \tag{2.4}$$

Here and below, by default $\boldsymbol{a} = \dot{\boldsymbol{u}}_T$, and $f = f(\dot{\boldsymbol{u}}_T)$ is the coefficient of friction.

Coulomb's simple law in displacements is acceptable only under simple loading,[15] so that in the general case the law (2.4) is necessary.

Under deformation, the contact of sets Γ_3 and Γ_4 occurs, so that

$$\{\mathbf{R}, \mathbf{N}, \mathbf{t}_n\}|\Gamma_3 = \{\mathbf{R}, -\mathbf{N}, -\mathbf{t}_n\}|\Gamma_4 \tag{2.5}$$

The vector $\mathbf{t}_T$ on sets Γ_3 and Γ_4 satisfies formulae of the type of (2.4).

The deformation will be unique if the following conditions[10] of retention of orientation and internal injectivity are imposed

$$\lambda(\mathbf{u}) = \det\mathbf{D}(\mathbf{r} + \mathbf{u}) > 0 \text{ allmost everywher in } \Omega, \quad \int\lambda(\mathbf{u})d\Omega \leq \operatorname{mes}\mathbf{R}(\Omega) \tag{2.6}$$

The classical problem with friction (2.1)–(2.6) (Problem PC) defines the classical solution $\mathbf{u}$. In the quasi-static case, slow processes are considered, so that the squares of the velocities and the acceleration are negligibly small, the loading parameter plays the role of 'slow' time and the presence of time reflects the influence of prehistory.

3. Some notation, spaces and results

Below, for spaces it is assumed that U^m is the m-fold direct product of spaces U with norm $||\cdot, U^m||$, $\mathbf{U} = U^3$ and $\tilde{\boldsymbol{U}} = \boldsymbol{U}_3$, and $U_1 \to (\mapsto) U_2$ is the continuous (compact) embedding of U_1 in U_2. Furthermore, $\to$ ($\tilde{\to}$) is strong (weak) convergence. And further, $|a| = (a * a)^{1/2}$, where $* = \cdot$ for a scalar or vector and $* = \cdot\cdot$ for a tensor. In the case of reflexivity or compactness, change to a subsequence, determined by the context, occurs by default.

As usual, $C_k(E)(C_k^0(E))$ is the space of k-fold continuously differentiable functions with compact support in the set E, and $L_p = L_p(\Omega)$, $1 \le p < \infty$, is the Lebesgue space of functions measurable and summable with power p on Ω. The Lebesgue integral and measure are used. The Sobolev space $W = W_p^1(\Omega)$ and Slobodetskii space (of traces) $V = W_p^{1-1/p}(\Gamma)$ are embedded in one another with $1 < p < \infty$,[16] so that

$$c_1\|u;\ V\| \le \|u;\ W\| \le c_2\|u;\ V\|\ \forall u \in W$$

Here

$$\|u;\ V\| = \{\int|u|^p d\Omega + \int|u(x) - u(y)|^p|x - y|^{-(p+1)}(d\Gamma_{(x)} \times d\Gamma_{(y)})\}^{1/p}$$

We will introduce classes of distribution functions[17,18]

$$g : t \in \Lambda \to U,\quad g \in G',\quad G = C_0^\infty(\Lambda)$$

where U is a Banach space, $\Lambda = [0, T]$ is the time interval, and $\dot{g}$ satisfies the equation

$$\int\dot{g}\varphi d\Lambda = -\int g\dot{\varphi}d\Lambda\ \forall\varphi \in G$$

We will introduce the Banach spaces

$$L = \{g|\|g(\cdot, t);\ U\| \in L_2(\Lambda)\},\quad W_{\Lambda, U} = W^2(\Lambda, U) = \{g|(g, \dot{g}, \ddot{g}) \in L^3\}$$
$$\mathbf{W}_0 = \{\mathbf{u} \in \mathbf{W}|\mathbf{u} = 0\ \text{ on }\ \Gamma_0\}$$

Since mes $\Gamma_0 > 0$, the norm $\left\|\boldsymbol{D}\boldsymbol{\upsilon};\ \vec{\boldsymbol{L}}_p\right\|$ is equivalent to the norm in $\mathbf{W}_0$.[19] From known relations[18] it follows that

$$W^2(\Lambda, U) \to C^1(\Lambda, U) = \{g(x, t)|g(x, \cdot) \in C^1(\Lambda), g(\cdot, t) \in U\}$$

The unit vectors of the outward normal to the boundary Γ_ψ and the boundary $\mathbf{R}(\Gamma_2)$ are equal to $\mathbf{n}_\psi = \mathbf{D}\psi/|\mathbf{D}\psi|$ and $\mathbf{N}_\psi(X) = -\mathbf{n}_\psi(\mathbf{R}(x))$ respectively. It can be assumed that $|\mathbf{D}\psi| \ge \delta \in R_+$. If $V_i = \{\upsilon \in V|\upsilon = 0$ outside $\Gamma_i\}$ and $\psi \in C^2(R^3)$, then from the inclusion $\mathbf{R} \in \mathbf{W}$ it follows that $\mathbf{N}_\psi \in \mathbf{V}_2$. Since $W \to C(\bar{\Omega})$ when $p > 3$, it follows that $u\upsilon \in W\ \forall(u,\ \upsilon) \in W^2$ (Ref. 10, Section 6.1) and

$$\|u\upsilon;\ W\| \le c\|u;\ W\|\|\upsilon;\ W\|\ \forall(u, \upsilon) \in W^2$$

By embedding theorems,[16] from the inclusions $\mathbf{u} \in \mathbf{W}$ and $\upsilon_N \in V_2$ it follows that $\upsilon_N = \upsilon_N\mathbf{N}_\psi \in \mathbf{V}_2$ and $|\upsilon_N| \in V_2$.

The following formulae of the tensor calculus are well known:

$$\nabla \cdot (\mathbf{Q} \cdot \mathbf{a}) = (\nabla \cdot \mathbf{Q}) \cdot \mathbf{a} + \mathbf{Q} \cdot\cdot \nabla \otimes \mathbf{a}^T,\quad \int\nabla \cdot \mathbf{Q}d\Omega = \int\mathbf{n} \cdot \mathbf{Q}d\Gamma$$

where the superscript T denotes transposition, and $\mathbf{n}$ is the unit vector of the outward normal to the boundary Γ.[12] Using these formulae and embedding theorems,[16,19] the following lemma is proved.

Lemma 3.1. *If Ω is the region defined above, $1 < p < \infty$,*

$$q = p/(p-1),\quad \mathbf{w} \in \mathbf{W},\quad \mathbf{T}(\mathbf{u}) \in \vec{\mathbf{L}}_q,\quad \nabla \cdot \mathbf{T} \in \mathbf{L}_1$$

then the tensor $\mathbf{T}$ has the trace $\mathbf{t}_n = \mathbf{T} \cdot \mathbf{n} \in \mathbf{V}'$, defined as a distribution by Green's formula

$$\int\mathbf{t}_n \cdot \mathbf{w}d\Gamma = \int(\nabla \cdot \mathbf{T}) \cdot \mathbf{w}d\Omega + \int\mathbf{T} \cdot\cdot \nabla \otimes \mathbf{w}^T d\Omega\ \forall\mathbf{w} \in \mathbf{W} \tag{3.1}$$

The function $g{:}\Omega \times R^m \to R$ possesses a Carathéodory property[20,21] (which is denoted as $g \in CAR$) if the function $g(x, \cdot)$ is continuous almost everywhere in Ω, while $g(\cdot, \xi)$ is measurable $\forall\xi \in R^m$. The function $h{:}u(x) \in S \to hu(x) = g(x, u(x)) \in R$, where S is the set of measurable functions from Ω in R^m, is the Nemytskii operator.[20] If

$$g : \Omega \times R^m \to R,\quad m \ge 1,\quad g \in CAR\ \text{ and }\ r \ge 1,\ p \ge 1$$

then $h{:}\mathbf{u} \in (L_p)^m \to h\mathbf{u} \in L_r$ is a continuous[20,21] and even a bounded operator,[21] i.e., it converts any bounded set into a bounded one.

For the functions $\ni(x, \mathbf{DR}(x))$ and $\mathbf{T}(\mathbf{u})$, the following two lemmas hold.

Lemma 3.2. *The function $\ni(x, \mathbf{D}(\mathbf{r}(x) + \mathbf{u}(x))) \in L_1$ if*

$$\beta \in\ ]-\infty, \infty[,\quad \beta \le \ni(x, \xi) \in C^1(\Omega \times R^9),\quad \mathbf{u} \in \mathbf{W},\quad 1 < p < \infty \tag{3.2}$$

Lemma 3.3. *If condition (3.2) is satisfied and $q = p/(p = 1)$, then $\mathbf{T}(\mathbf{u}) \in \vec{\mathbf{L}}_q$.*

Proof. Generally speaking,

$$t^{-1}[\ni(x, \mathbf{DR} + t\mathbf{D}\boldsymbol{\upsilon}) - \ni(x, \mathbf{DR})] = \ni_\xi(x, \mathbf{DR} + \theta\mathbf{D}\boldsymbol{\upsilon}) \cdot\cdot \mathbf{D}\boldsymbol{\upsilon}^T$$

according to Lagrange's formula, where $\theta \in (0, t) \subset R_+$. According to Lemma 3.2, the left-hand side of this equation belongs to L_1, and consequently so does the right-hand side. The right-hand side converges to $g(\mathbf{u}, \upsilon) = \Im_\xi(x, \mathbf{DR})\,\mathbf{D}\upsilon^T$ almost everywhere in Ω with $t \to 0$. By Fatou's lemma, $g \in L_1\ \forall(\mathbf{u}, \upsilon) \in \mathbf{W}^2$, so that the operator $g(\mathbf{u}, \upsilon)$ from $\mathbf{W}^2$ in L_1 exists and is continuous and bounded, with the bounded norm

$$\|\mathbf{T}(\mathbf{u})\| = \sup\{\textstyle\int g(\mathbf{u}, \boldsymbol{\upsilon})d\Omega \,|\, \|\boldsymbol{\upsilon}; \mathbf{W}\| = 1\}.$$

4. Variational formulation of the problem

If $\upsilon \in C^2(\Lambda, \mathbf{W})$, then $\upsilon \in \mathbf{C}(\Omega)$. We further assume that

$$s = t + dt > t \ \text{и}\ \Gamma_{4+k}^t = \lim_{s \to t}(\Gamma_k^t \cap \Gamma_k^s), \quad k = 2, 3, 4$$

Then

$$\Gamma_6^t = \{x|\psi(\mathbf{R}(x, t)) = 0, \dot{u}_N(x, t) = 0\}, \quad (\mathbf{R}, \dot{\mathbf{R}})\big|\Gamma_7 = (\mathbf{R}, \dot{\mathbf{R}})\big|\Gamma_8$$

From the non-penetration condition $\psi(\mathbf{r}(x)) + \upsilon(x, \tau) \geq 0$ for $\tau = t, s$, where $s \to t$, it follows that $\dot{\upsilon}_N \leq 0$ on Γ_6^t. As the set $C^2(\Lambda, U)$ is dense in $W^2(\Lambda, U)$, it is possible to introduce the sets

$$\mathbf{K}_{7,8}^t = \{\boldsymbol{\upsilon}(x, t) \in \mathbf{W}|\mathbf{R}_{\boldsymbol{\upsilon}}(\Gamma_7) = \mathbf{R}_{\boldsymbol{\upsilon}}(\Gamma_8)\}$$

$$\mathbf{K}_\lambda^t = \{\boldsymbol{\upsilon}(x, t) \in \mathbf{W}|\lambda(\boldsymbol{\upsilon}) > 0 \text{ allmost everywhere in } \Omega, \textstyle\int\lambda(\boldsymbol{\upsilon})d\Omega \leq \operatorname{mes}\mathbf{R}_{\boldsymbol{\upsilon}}(\Omega)\}$$

$$\mathbf{K}^t = \{\boldsymbol{\upsilon}(x, t) \in \mathbf{W} \cap \mathbf{K}_\lambda^t \cap \mathbf{K}_{7,8}^t\big|\psi(\mathbf{R}_{\boldsymbol{\upsilon}}) \geq 0 \text{ on } \Gamma_6^t\}$$

$$\mathbf{K} = \mathbf{K}(\mathbf{u}) = \{\boldsymbol{\upsilon} \in W_{\Lambda,\mathbf{W}} \cap \mathbf{K}^t\big|\dot{\boldsymbol{\upsilon}} \in \mathbf{K}_{7,8}^t, \dot{\upsilon}_N \leq 0 \text{ on } \Gamma_6^t\}$$

We will introduce the scalar products $]\cdot, \cdot[$, $\langle\cdot, \cdot\rangle$ and $[\cdot, \cdot]$ into $L_2(\Gamma)$, $L_2(\Gamma_5)$ and $L_2(\Gamma_6)$; $(\mathbf{a}, \mathbf{b}) = \int \mathbf{a} * \mathbf{b}\,d\Omega$, the set $\Gamma_9 = \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$ and the variation of the displacement $\delta\mathbf{u} = \upsilon - \mathbf{u}$, where $\upsilon \in \mathbf{K}$. assume that

$$p > 3, \quad \rho\mathbf{S} \in \mathbf{W}_0', \quad \mathbf{P}_n \in \mathbf{V}_1', \quad \mathbf{F} \in \vec{\mathbf{L}}_\infty(\Gamma_9) \tag{4.1}$$

We multiply the first equation of system (2.1) by $\delta\dot{\mathbf{u}}$ and integrate with respect to Ω using relation (3.1). We obtain

$$(\mathbf{T}, \mathbf{D}\delta\dot{\mathbf{u}}^T) + (\rho\ddot{\mathbf{u}} - \rho\mathbf{S}, \delta\dot{\mathbf{u}}) -]\mathbf{t}_n, \delta\dot{\mathbf{u}}[= 0 \tag{4.2}$$

We will prove that

$$H = \mathbf{t}_T \cdot (\dot{\boldsymbol{\upsilon}}_T - \dot{\mathbf{u}}_T) + |t_\mathbf{f}|(|\dot{\boldsymbol{\upsilon}}_\mathbf{f}| - |\dot{\mathbf{u}}_\mathbf{f}|) \geq 0 \text{ on } \Gamma_5$$

If $|\mathbf{t}_T| < |t_\mathbf{f}|$ and $\dot{\mathbf{u}}_T = 0$, then

$$H = \mathbf{t}_T \cdot \dot{\boldsymbol{\upsilon}}_\mathbf{f} + |t_\mathbf{f}||\dot{\boldsymbol{\upsilon}}_\mathbf{f}| \geq 0$$

If, however, $\mathbf{t}_T = |t_N|\mathbf{f}$, then

$$-\mathbf{t}_T \cdot \dot{\mathbf{u}}_T = (\mathbf{l} \cdot \mathbf{F} \cdot \mathbf{l})|t_N||\dot{\mathbf{u}}_T| = -|t_N|\mathbf{f} \cdot \dot{\mathbf{u}}_T = -|t_\mathbf{f}|\dot{\mathbf{u}}_\mathbf{f} \geq 0$$

since the tensor $\mathbf{F}$ is positive-definite, and therefore

$$-\mathbf{t}_T \cdot \dot{\mathbf{u}}_T - |t_\mathbf{f}|\dot{\mathbf{u}}_\mathbf{f} = 0, \quad \text{and} \quad H \geq 0$$

We will define

$$L(\mathbf{w}) = (\rho\mathbf{S}, \mathbf{w}) + \int(\mathbf{P}_n \cdot \mathbf{w})d\Gamma_1$$

In the quasi-stationary case it is possible to drop $\rho\ddot{\mathbf{u}}$. Using expansions for $\mathbf{t}_n$ and $\delta\dot{\mathbf{u}}$, relations (2.2) and (2.5) and the conditions that $H \geq 0$ on Γ_5 and $t_N\delta\dot{u}_N \geq 0$ on Γ_6, from Eq. (4.2) we obtain

$$(\mathbf{T}, \mathbf{D}\delta\dot{\mathbf{u}}^T) - L(\delta\dot{\mathbf{u}}) - \langle t_\mathbf{f}, |\dot{\boldsymbol{\upsilon}}_\mathbf{f}| - |\dot{\mathbf{u}}_\mathbf{f}|\rangle \geq 0 \ \forall\delta\dot{\mathbf{u}}, \boldsymbol{\upsilon} \in \mathbf{K} \tag{4.3}$$

Note that, in the isotropic case,

$$\dot{\boldsymbol{\upsilon}}_\mathbf{f} = -\dot{\boldsymbol{\upsilon}}_T, \quad \dot{\mathbf{u}}_\mathbf{f} = -\dot{\mathbf{u}}_T$$

It is natural to determine the weak solution $\mathbf{u}(x, t)$ on the basis of inequality (4.3). Then

$$\mathbf{u}(x,t) \in \mathbf{W}, \quad \mathbf{T}(\mathbf{u}) \in \vec{\mathbf{L}}_q$$

according to Lemma 3.3, and $\mathbf{t}_n$ and t_N are distributions on the boundary, and here $t_N \le 0$ on Γ_6. As a result, taking into account the second equation of system (2.1) and Eq. (3.1), we have

$$\mathbf{T} = \ni_\xi(x, \mathbf{DR}), \quad 0 \le [t_N, \upsilon_N] = (\mathbf{T}, \mathbf{D}\boldsymbol{\upsilon}_N^T) - (\rho\mathbf{S}, \boldsymbol{\upsilon}_N) \;\; \forall \boldsymbol{\upsilon}_N \in \mathbf{V}_6 \tag{4.4}$$

If $\upsilon_N \in V_6$, then $\upsilon_N \in V_6$ and is extended continuously in $\mathbf{W}$. Relation (3.1), the Hölder inequality and the embedding theorems indicate that $t_N \in V'_6$. Since $\left|\dot{\boldsymbol{\upsilon}}\right| \in \mathbf{V}$ and $\Gamma_5^t \subset \Gamma_6^t$, it follows that $t_N \in V'_5$ and $ft_N \in V'_5$, and the functional $\left\langle t_\mathbf{f}, \; \left|\dot{\boldsymbol{\upsilon}}_\mathbf{f}\right| - \left|\dot{\mathbf{u}}_\mathbf{f}\right|\right\rangle$ is determined. The weak problem (PW) means finding $\mathbf{u}(x, t) \in \mathbf{K}$ from relations (4.3) and (4.4). The equivalence of auxiliary problems when certain additional conditions are observed is proved as usual,[6,8,9,22] and this can be dropped. If conditions (3.2) and (4.1) are satisfied, then Problem PC is equivalent to Problem PW.

Following Kravchuk,[9] we expand the left-hand side of inequality (4.3) for the instant of time s in terms of $dt \le dt_*$ with fairly small dt_*. We will introduce the functions

$$\tilde{\boldsymbol{\upsilon}}(x,t) = \boldsymbol{\upsilon}(x,s) - \boldsymbol{\upsilon}(x,t) + \mathbf{u}^t, \quad \mathbf{u}^t = \mathbf{u}(x,t), \quad \tilde{\mathbf{u}}(x,t) = \mathbf{u}(x,s), \quad \tilde{\mathbf{w}} = \tilde{\boldsymbol{\upsilon}} - \tilde{\mathbf{u}}$$

and drop powers below dt^2. As a result, we obtain

$$(\mathbf{T}, \mathbf{D}\tilde{\mathbf{w}}^T) - L(\tilde{\mathbf{w}}) - \langle t_\mathbf{f}, \left|\boldsymbol{\upsilon}_\mathbf{f} - \mathbf{u}_\mathbf{f}^t\right| - \left|\tilde{\mathbf{u}}_\mathbf{f} - \mathbf{u}_\mathbf{f}^t\right|\rangle \ge 0 \;\; \forall \tilde{\boldsymbol{\upsilon}} \in \tilde{\mathbf{K}} \tag{4.5}$$

Here

$$\tilde{\mathbf{u}} \in \tilde{\mathbf{K}} = \{\tilde{\mathbf{h}} | \mathbf{h}(x,\tau) \in \mathbf{K}^\tau, \tau = t, s, \quad h_N(s) \le h_N(t) \text{ on } \Gamma_6\}, \quad \mathbf{a} = \tilde{\mathbf{u}} - \mathbf{u}^t$$

Suppose conditions (3.2) and (4.1) are satisfied. We will formulate Problem PW^t: it is required to find $\tilde{\mathbf{u}}$ from relations (4.5) and (4.4). Problems PW^t and PW are equivalent if dt is sufficiently small. Applying the iteration method[4,9] to Problem PW^t, we obtain Problem PW$^{(k)}$ with "specified friction" (static, if $\mathbf{u}_\mathbf{f}^t$ is removed): it is required to find $\tilde{\mathbf{u}}_{(k+1)} \in \tilde{\mathbf{K}}$ and $t_N^{(k+1)}$ with $\mathbf{w} = \tilde{\boldsymbol{\upsilon}} - \tilde{\mathbf{u}}_{(k+1)} \; \forall \tilde{\boldsymbol{\upsilon}} \in \tilde{\mathbf{K}}$ and $\mathbf{a} = \tilde{\mathbf{u}}_{(k)} - \mathbf{u}_t$ from the system

$$(\mathbf{T}^{(k+1)}, \mathbf{D}\tilde{\mathbf{w}}^T) - L(\tilde{\mathbf{w}}) \ge \langle t_\mathbf{f}^{(k)}, \left|\tilde{\boldsymbol{\upsilon}}_\mathbf{f} - \mathbf{u}_\mathbf{f}^t\right| - \left|\tilde{\mathbf{u}}_\mathbf{f}^{(k+1)} - \mathbf{u}_\mathbf{f}^t\right|\rangle \tag{4.6}$$

$$\mathbf{T}^{(k+1)} = \ni_\xi(x, \mathbf{D}(\mathbf{r} + \tilde{\mathbf{u}}^{(k+1)})) \tag{4.7}$$

$$[t_N^{(k+1)}, \upsilon_N] = (\mathbf{T}^{(k+1)}, \mathbf{D}\boldsymbol{\upsilon}_N^T) - (\rho\mathbf{S}, \boldsymbol{\upsilon}_N) \quad \forall \upsilon_N \in V_6 \tag{4.8}$$

5. The problem for specified friction

For $p > 3$, $q = p/2$ and $r = p/3$, we will introduce the spaces

$$\mathbf{G} = \vec{\mathbf{L}}_q \times L_r, \quad \mathbf{L} = \mathbf{L}_p \times \mathbf{G}, \quad \mathbf{B} = \mathbf{W} \times \mathbf{G}$$

and operators

$$\tilde{\lambda}(\boldsymbol{\upsilon}) = \det\mathbf{D}\boldsymbol{\upsilon}, \quad \chi(\boldsymbol{\upsilon}) = \mathrm{cof}(\mathbf{DR}_\boldsymbol{\upsilon}), \quad \tilde{\chi}(\boldsymbol{\upsilon}) = \mathrm{cof}\mathbf{D}\boldsymbol{\upsilon}, \quad A\boldsymbol{\upsilon} = (\chi(\boldsymbol{\upsilon}), \lambda(\boldsymbol{\upsilon}))$$

$$\tilde{A}\boldsymbol{\upsilon} = (\boldsymbol{\upsilon}, A\boldsymbol{\upsilon})$$

where $\mathrm{cof}\mathbf{H} = (\det\mathbf{H})\mathbf{H}^{-T}$. Well-known results[10] give the following.

Lemma 5.1. *The operator A:$\mathbf{W} \to \mathbf{G}$ is weakly and strongly continuous, closed and bounded. Then*

$$\|\tilde{\lambda}(\boldsymbol{\upsilon})\| \le c_1 \|\boldsymbol{\upsilon}\|^3, \quad \|\tilde{\chi}(\boldsymbol{\upsilon})\| \le c_2 \|\boldsymbol{\upsilon}\|^2$$

and consequently

$$\|\lambda(\boldsymbol{\upsilon})\| \le P_3(\|\boldsymbol{\upsilon}\|), \quad \|\chi(\boldsymbol{\upsilon})\| \le P_2(\|\boldsymbol{\upsilon}\|)$$

where P_2 and P_3 are polynomials of degree 2 and 3 respectively, positive on R_+, and the norms are taken for λ and $\tilde{\lambda}$ in L_r, for χ and $\tilde{\chi}$ in $\vec{\mathbf{L}}_q$ and for υ in $\mathbf{W}_0$.

Lemma 5.2. *The sets $\mathbf{K}_{7,8}$, $\mathbf{K}_\lambda$, $\mathbf{K}^t$ and $\tilde{\mathbf{K}}$ are weakly closed.*

Proof. Suppose $\tilde{\boldsymbol{\upsilon}}_n \in \tilde{\mathbf{K}}$ and $\tilde{\boldsymbol{\upsilon}}_n \tilde{\rightarrow} \tilde{\boldsymbol{\upsilon}}_0$ in $\mathbf{W}_0$. Since $\boldsymbol{\upsilon}_n(x, \;\tau) \tilde{\rightarrow} \boldsymbol{\upsilon}_0(x, \;\tau)$ with $\tau = t, s$ and $\mathbf{W} \mapsto \mathbf{C}(\Gamma)$, it follows that $\upsilon_n \to \upsilon_0$ in $\mathbf{C}(\Gamma)$. This means that $\upsilon_{0,N}(s) \le \upsilon_{0,N}(t)$ on Γ_6, $\upsilon_0 \in \mathbf{K}_{7,8}$ and $\psi(\mathbf{r} + \upsilon_0) \ge 0$ on Γ_6^τ. Then, with the usual reasoning (Ref. 10, Sections 7.7 and 7.9), we conclude that $\upsilon_0 \in \mathbf{K}_\lambda$. Consequently, $\upsilon_0 \in \mathbf{K}^\tau$ and $\tilde{\boldsymbol{\upsilon}}_0 \in \tilde{\mathbf{K}}$.

Lemma 5.3. *If $\tilde{\mathbf{u}} \in \tilde{\mathbf{K}}$ and $\tilde{\boldsymbol{\upsilon}} \in \tilde{\mathbf{K}}$, then in the set $\tilde{\mathbf{K}}$ there will be the sequence $\{\tilde{\mathbf{u}}_{(n)}\}$ such that $\tilde{\mathbf{u}}_{(n)} - \tilde{\mathbf{u}} \to 0$ and $n(\tilde{\mathbf{u}}_{(n)} - \tilde{\mathbf{u}}) \to \tilde{\boldsymbol{\upsilon}} - \tilde{\mathbf{u}}$ in $\mathbf{W}$ as $n \to \infty$.*

Proof. Let

$$\mathbf{u}^{(n)} \equiv \mathbf{h} = \mathbf{u} + n^{-1}\mathbf{w} \in \mathbf{W}_0, \quad \mathbf{w} = \boldsymbol{\upsilon} - \mathbf{u} + n^{-1/2}\mathbf{g}$$

and here the smooth function $\mathbf{g}$ is chosen such that $(-w_N) \geq \delta \in R_+$ and $\dot{\mathbf{g}} = 0$ in Ω. Obviously, $\mathbf{u}^{(n)} - \mathbf{u} \to 0$ and $n(\mathbf{u}^{(n)} - \mathbf{u}) \to \upsilon - \mathbf{u}$ in $\mathbf{W}$, and $\mathbf{W} \to \mathbf{C}(\Gamma)$. Since

$$|\mathbf{w}| < c_1, \quad |\mathbf{D}\psi(\mathbf{R})| > c_2, \quad O(n^{-2}|\mathbf{w}|^2) < c_3 n^{-2} \text{ on } \Gamma_6^\tau, \quad c_i \in R_+, \quad i = 1, 2, 3$$

$$\psi(\mathbf{R_h}) = \psi(\mathbf{R}) + \mathbf{D}\psi(\mathbf{R}) \cdot n^{-1}\mathbf{w} + O(|n^{-1}\mathbf{w}|^2) \geq c_3 n^{-3/2} d(n)$$

where the function $d(n) > 0$ when $n > n_0 = c_3^2(\delta c_2)_{-2}$, it follows that

$$\psi(\mathbf{R_h}) \geq 0 \text{ in } \Omega \text{ when } n \geq n_0$$

We will use Minkowski's inequality for the determinants

$$\lambda(\mathbf{h}) = \lambda((1-n^{-1})\mathbf{u} + n^{-1}\boldsymbol{\upsilon} + n^{-1}\mathbf{g}) \geq (1-n^{-1})^3\lambda(\mathbf{u}) + n^{-3}\lambda(\boldsymbol{\upsilon}) + n^{-3}\lambda(\mathbf{g})$$

Almost everywhere in Ω we have $\lambda(\mathbf{u}) > 0$ and $\lambda(\upsilon) > 0$, so that a number n_1 will be found such that $\lambda(\mathbf{u}^{(n)}) > 0$ almost everywhere in Ω with $n > n_1$.Since

$$\det(\mathbf{A} + \mathbf{B}) = \det\mathbf{A} + \operatorname{cof}\mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot\cdot \operatorname{cof}\mathbf{B} + \det\mathbf{B}$$

it follows that

$$\lambda(\mathbf{h}) = \lambda(\mathbf{u}) + n^{-1}\chi(\mathbf{u}) \cdot\cdot \mathbf{D}\mathbf{w} + n^{-2}\mathbf{D}\mathbf{R} \cdot\cdot \tilde{\chi}(\mathbf{w}) + n^{-3}\tilde{\lambda}(\mathbf{w})$$

We use Hölder's inequalities and Lemma 5.1. Then

$$\int\lambda(\mathbf{h})d\Omega \leq \int\lambda(\mathbf{u})d\Omega + n^{-1}\Phi(\|\mathbf{u}\|, \|\mathbf{w}\|)$$

where $\Phi \geq 0$ is a bounded function and $||\cdot|| = ||\cdot;\ \mathbf{W}_0||$. For any $\varepsilon > 0$, a number n_2 will be found such that

$$\int\lambda(\mathbf{h})d\Omega \leq \int\lambda(\mathbf{u})d\Omega + \varepsilon/2 \qquad \forall n \geq n_2$$

Since $\mathbf{W} \mapsto \mathbf{C}(\bar{\boldsymbol{\Omega}}) \Rightarrow \mathbf{u}_{(n)} \to \mathbf{u}$ in $\mathbf{C}(\bar{\Omega})$, a number $n_3 \geq n_2$ will be found (Ref. 10, Section 7.9) such that

$$\operatorname{mes}\mathbf{R}(\Omega) \leq \operatorname{mes}\mathbf{R_h}(\Omega) + \varepsilon/2 \quad \forall n \geq n_2, \quad \int\lambda(\mathbf{u}^{(n)})d\Omega \leq \int\lambda(\mathbf{u})d\Omega + \varepsilon/2 \leq \operatorname{mes}\mathbf{R}(\Omega) + \varepsilon/2 \leq \operatorname{mes}\mathbf{R_h}(\Omega) + \varepsilon$$

Since the quantity ε is arbitrary,

$$\int\lambda(\mathbf{u}^{(n)})d\Omega \leq \operatorname{mes}\mathbf{R_h}(\Omega)$$

for fairly large n. As a result, $\mathbf{u}_{(n)} \in \mathbf{K}_\lambda$ and $\mathbf{u}^{(n)} \in \mathbf{K}^\tau$. The fact that $\tilde{\mathbf{u}}$ and $\tilde{\boldsymbol{\upsilon}} \in \tilde{\mathbf{K}}$ indicates that $u_N(s) \leq u_N(t)$ and $\upsilon_N(s) \leq \upsilon_N(t)$ on Γ_6^t. Hence, taking into account that $g_N(s) = g_N(t)$, we obtain $u_N^{(n)}(s) \leq u_N^{(n)}(t)$ on Γ_6^t and $\mathbf{u}_{(n)} \in \tilde{\mathbf{K}}$.

The set $\tilde{\mathbf{K}}$ is non-convex, and the change from Problem $PW^{(k)}$ to the problem of minimizing the functional is difficult. We will introduce as usual[17] the set $E(\tilde{\mathbf{K}}, \tilde{\mathbf{u}}) = \{\tilde{\mathbf{w}} \in \mathbf{W}_0\}$, for each element of which elements $\tilde{\mathbf{u}}_{(n)} \in \tilde{\mathbf{K}}$ and $\mu_n \in R_+$ exist such that $\tilde{\mathbf{u}}_{(n)} \to \tilde{\mathbf{u}}$ and $\mu_n(\tilde{\mathbf{u}}(n) - \tilde{\mathbf{u}}) \to \tilde{\mathbf{w}}$ in $\mathbf{W}_0$. Since, according to Lemma 5.3, from the inclusion $(\tilde{\mathbf{u}}, \tilde{\boldsymbol{\upsilon}}) \in \tilde{\mathbf{K}}_2$ it follows that $\tilde{\boldsymbol{\upsilon}} - \tilde{\mathbf{u}} \in E$, the following theorem is applicable, the proof of which, (similar to the case $J_2 = 0$[17]) is omitted.

Theorem 5.1. *If $\tilde{\mathbf{u}} \in \tilde{\mathbf{K}}$, and $J(\tilde{\mathbf{u}}) \leq J(\tilde{\boldsymbol{\upsilon}})\ \forall \tilde{\boldsymbol{\upsilon}} \in \tilde{\mathbf{K}}$, where $J = J_1 + J_2$, J_1 is differentiated and J_2 is a continuous functional on $\tilde{\mathbf{K}}$, then*

$$\mathbf{D}J_1(\tilde{\mathbf{u}}) \cdot (\tilde{\boldsymbol{\upsilon}} - \tilde{\mathbf{u}}) + J_1(\tilde{\boldsymbol{\upsilon}}) - J_2(\tilde{\mathbf{u}}) \geq 0 \ \forall \tilde{\boldsymbol{\upsilon}} - \tilde{\mathbf{u}} \in E(\tilde{\mathbf{K}}, \tilde{\mathbf{u}})$$

We will formulate the Problem PM:

$$\tilde{\mathbf{u}}^{(k+1)} \to \inf\{J(\tilde{\boldsymbol{\upsilon}}) | \tilde{\boldsymbol{\upsilon}} \in \tilde{\mathbf{K}}\}$$

$$J = J_1 + J_2$$

$$J_1(\tilde{\boldsymbol{\upsilon}}) = \int \mathrm{Э}(x, \mathbf{D}(\mathbf{r} + \tilde{\boldsymbol{\upsilon}}))d\Omega - L(\tilde{\boldsymbol{\upsilon}}), \quad J_2(\tilde{\boldsymbol{\upsilon}}) = \langle t_{\mathbf{f}}^{(k)}, |\tilde{\boldsymbol{\upsilon}}_{\mathbf{f}} - \mathbf{u}_{\mathbf{f}}^t|\rangle, \quad \mathbf{a} = \tilde{\mathbf{u}}^{(k)} - \mathbf{u}^t$$

Minimization of the functional J is problematical, since the function $\mathrm{Э}(x, \mathbf{H})$ is not convex with respect to $\mathbf{H}$. Ball[10,11] introduced a weaker condition of polyconvexity. Suppose $M^0 \subset M = \{\mathbf{H} \in R^{3\times 3}\}$, where $\mathbf{H}$ is a matrix and $\operatorname{Co}\hat{M}_0$ is the convex shell of the set

$$\hat{M}^0 = \{\hat{\mathbf{H}} = (\mathbf{H}, \operatorname{cof}\mathbf{H}, \det\mathbf{H}) | \mathbf{H} \in M^0\}$$

The function $\ni(x,\cdot)$: $M^0 \to R$ is polyconvex if a convex function $\tilde{\ni}(x,\cdot) : \mathrm{Co}\hat{M}^0 \to R$ exists such that

$$\ni(x, H) = \tilde{\ni}(x, \hat{\mathbf{H}})\ \forall \mathbf{H} \in M^0 \text{ allmost everywhere in } \Omega.$$

Theorem 5.2. *Suppose conditions (3.2) and (4.1) are satisfied, and, furthermore, that:*

(1) $\psi \in C^2(R^3)$;
(2) *the function* $\ni(x,\cdot) : \mathrm{Co}\mathbf{D}(\tilde{\mathbf{K}}) \to R$ *is poly-convex according to Ball;*
(3) *the condition of coercivity is satisfied: constants* α *and* β, $\alpha > 0$ *exist, such that*

$$\ni(x, \mathbf{H}) \geq \alpha(|\mathbf{H}|^p + |\mathrm{cof}\mathbf{H}|^q + |\det\mathbf{H}|^r) + \beta \quad \forall \mathbf{H} \in M, \quad \text{where} \quad r > 1 \tag{5.1}$$

(4) $\ni(x, \mathbf{H}) \to +\infty$ *almost everywhere in* Ω *as* $\det \mathrm{H} \to +0$;
(5) $\inf\{J(\tilde{\boldsymbol{\upsilon}}) \mid \tilde{\boldsymbol{\upsilon}} \in \tilde{\mathbf{K}}\} < +\infty$;
(6) $J_2(\tilde{\boldsymbol{\upsilon}}) \leq a\| t_{\mathbf{f}}^{(k)}; V_6'\| (\||\tilde{\boldsymbol{\upsilon}}|; V_6\| + b)$, *where* $\tilde{\boldsymbol{\upsilon}} \in \tilde{\mathbf{K}}$, $a, b \in R_+$.

A solution of Problem PM then exists, and this problem is equivalent to Problem PW$^{(k)}$.

Proof. We will use Ball's scheme for the problem without friction.[10,11] If $\tilde{\boldsymbol{\upsilon}} \in \mathbf{W}_0$, then $\tilde{A}\tilde{\boldsymbol{\upsilon}} \in \mathbf{B}$. Taking into account the last condition of the theorem, the continuity and boundedness of the operators $\ni(x,\cdot) : \vec{L}_p \to L_1$ and $\tilde{\ni}(x,\cdot) : \mathbf{B} \to L_1$ and also the inequality

$$|\mathbf{D}\tilde{\boldsymbol{\upsilon}}|^p \leq 2^{p-1}(|\mathbf{D}(\mathbf{r} + \tilde{\boldsymbol{\upsilon}})|^p + 3^{p/2})$$

we integrate inequality (5.1). As a result, we obtain

$$J(\tilde{\boldsymbol{\upsilon}}) \geq c_1\|\tilde{A}\tilde{\boldsymbol{\upsilon}}; \mathbf{B}\|^l + c_2, \quad l = \inf(p, r), \quad c_1 > 0 \tag{5.2}$$

Suppose $\{\tilde{\boldsymbol{\upsilon}}_m\} \in \tilde{\mathbf{K}}$ and $J(\tilde{\boldsymbol{\upsilon}}_m) \to \inf\{J(\tilde{\boldsymbol{\upsilon}}) \mid \tilde{\boldsymbol{\upsilon}} \in \tilde{\mathbf{K}}\}$ as $m \to \infty$. On account of condition 5, inequality (5.2), the reflexivity of space $\mathbf{B}$ and Lemma 5.1, $\tilde{A}\tilde{\boldsymbol{\upsilon}}_n \tilde{\to} \tilde{A}\tilde{\boldsymbol{\upsilon}}_0$ exists in $\mathbf{B}$. Then, $\tilde{\boldsymbol{\upsilon}}_n \tilde{\to} \tilde{\boldsymbol{\upsilon}}_0$, and, when $\tau = t, s$, there will be $\tilde{\boldsymbol{\upsilon}}_n(x,\tau) \tilde{\to} \tilde{\boldsymbol{\upsilon}}_0(x,\tau)$ in $\mathbf{W}_0$. Since $\mathbf{W} \mapsto \mathbf{C}(\Gamma)$, it follows that $\boldsymbol{\upsilon}_n(x,\tau) \to \boldsymbol{\upsilon}_0(x,\tau)$ uniformly on Γ_6^τ and Γ_0. This means that

$$\upsilon_{0,N}(s) \leq \upsilon_{0,N}(t) \text{ on } \Gamma_6^t, \quad \tilde{\boldsymbol{\upsilon}}_0 = 0 \text{ on } \Gamma_0, \quad \psi(\mathbf{r} + \boldsymbol{\upsilon}_0(x,\tau)) \geq 0 \text{ on } \Gamma_6^\tau$$

According to Lemma 5.2, $\upsilon_0 \in \mathbf{K}_\lambda$. Consequently, $\upsilon_0 \in \mathbf{K}^\tau$ and $\tilde{\boldsymbol{\upsilon}}_0 \in \tilde{\mathbf{K}}$. With the usual reasoning (Ref. 10, Section 7.7), we find that $J(\tilde{\boldsymbol{\upsilon}}_0) = \inf\{J(\tilde{\boldsymbol{\upsilon}}) \mid \tilde{\boldsymbol{\upsilon}} \in \tilde{\boldsymbol{K}}\}$. The theorem is proved.

Thus, for Problem PM we have

$$\tilde{\mathbf{u}}^{(k+1)}(t_{\mathbf{f}}^{(k)}) \in \mathbf{W}_0, \quad \mathbf{T}^{(k+1)} = \mathbf{T}(\tilde{\mathbf{u}}^{(k+1)}) \in \vec{\mathbf{L}}_q, \quad q = p/(p-1)$$

$$t_N^{(k+1)} = t_N(\mathbf{T}^{(k+1)}) \in V_6'$$

Hence, $t_{\boldsymbol{f}}^{(k+1)} \in V_6'$, and in the topology V'_6 the operator $Q : t_{\boldsymbol{f}}^{(k)} \to t_{\boldsymbol{f}}^{(k+1)}$ is determined. It is not difficult to prove that, on Γ_6^t, from $t_N^{(k)} \leq 0$ it follows that $t_N^{(k+1)} \leq 0$.

6. Application of the fixed-point principle

It follows from Section 5 that the following problem PW$^{(k)}$ is solvable: it is required to find $\tilde{\mathbf{u}}_{(k+1)} \in \tilde{\mathbf{K}}$ from relations (4.6) to (4.8).

Lemma 6.1. *The operator Q is weakly continuous in the topology* V'_6.

Proof. Suppose

$$J(\tilde{\boldsymbol{\upsilon}}, t_{\mathbf{f}}) = J_1(\tilde{\boldsymbol{\upsilon}}) + J_2(\tilde{\boldsymbol{\upsilon}}, t_{\mathbf{f}}), \quad J_2(\mathbf{w}, \mu) = -\langle \mu, |\mathbf{w}_{\mathbf{f}} - \mathbf{u}_{\mathbf{f}}^t| \rangle$$

and $\tilde{\mathbf{u}}_m$ and $\tilde{\mathbf{u}}$ are the solutions of problem PM for $t_{\mathbf{f}} = \mu_m$ and μ respectively. Then

$$J(\tilde{\mathbf{u}}_m, \mu_m) + J_2(\tilde{\mathbf{u}}_m, \mu - \mu_m) \geq J(\tilde{\mathbf{u}}, \mu) \geq J(\tilde{\mathbf{u}}_m, \mu_m) - J_2(\tilde{\mathbf{u}}, \mu_m - \mu)$$

From this we conclude that

$$-J_2(\tilde{\mathbf{u}}_m, \mu - \mu_m) \leq J(\tilde{\mathbf{u}}_m, \mu_m) - J(\tilde{\mathbf{u}}, \mu) \leq -J_2(\tilde{\mathbf{u}}, \mu - \mu_m) \tag{6.1}$$

$$J(\tilde{\mathbf{u}}_m, \mu_m) \leq J(0, \mu_m) = -\langle \mu_m, |\mathbf{u}_{\mathbf{f}}^t| \rangle \tag{6.2}$$

If $\mu_m \tilde{\to} \mu$ in V'_6, then, using the boundedness of the sequence $(\|\mu_m\|)$, the reflexivity of the space $\mathbf{B}$, relations (6.2) and (5.2) and the closedness of the operator A, we will single out $\tilde{A}\tilde{\mathbf{u}}_m \tilde{\to} \tilde{A}\tilde{\boldsymbol{\upsilon}}_0$. Since $\mathbf{W} \mapsto \mathrm{C}(\Gamma_6)$, it follows that $\tilde{\mathbf{u}}_m \to \tilde{\boldsymbol{\upsilon}}_0$ in $\mathbf{C}(\Gamma_6)$. This fact, inequality (6.1) and the fact that $\mu_m \tilde{\to} \mu$ give

$$\lim J(\tilde{\mathbf{u}}_m, \mu_m) = J(\tilde{\boldsymbol{\upsilon}}_0, \mu), \quad \lim J_2(\tilde{\mathbf{u}}_m, \mu_m) = J_2(\tilde{\boldsymbol{\upsilon}}_0, \mu) \quad \text{as} \quad m \to \infty$$

On account of the polyconvexity of the function Э and Mazur's and Fatou's lemmas, we have

$$J_1(\tilde{\mathbf{v}}_0) \leq \lim J_1(\tilde{\mathbf{u}}_m)$$

(Ref. 10, Section 7.7), so that

$$J(\tilde{\mathbf{v}}_0, \mu) \leq J(\tilde{\mathbf{u}}, \mu) = \inf\{J(\tilde{\mathbf{v}}, \mu) | \mathbf{v} \in \tilde{\mathbf{K}}\}$$

The contradiction is eliminated if $\tilde{\mathbf{v}}_0 = \tilde{\mathbf{u}}$ and $\tilde{\mathbf{u}}_m \tilde{\to} \tilde{\mathbf{u}}$ in the space $\mathbf{W}$.

From the results of Section 3 it follows that the operator $\mathbf{T} : \mathbf{W} \to \vec{\mathbf{L}}_q$ is continuous and bounded. Since $\tilde{\mathbf{u}}_m \tilde{\to} \tilde{\mathbf{u}}$ in $\mathbf{W}$ and the sequence $\{\mathbf{T}(\tilde{\mathbf{u}}_m)\}$ is bounded, it is possible to isolate $\mathbf{T}(\tilde{\mathbf{u}}_m) \tilde{\to} \mathbf{T}_*$ in $\vec{\mathbf{L}}_q$, where $q = p/(p-1)$. In inequality (4.6) defining $\tilde{\mathbf{u}}$ we will assume that $\tilde{\mathbf{v}} = \tilde{\mathbf{u}} \pm \boldsymbol{\varphi}$, and in equality (4.6) defining $\tilde{\mathbf{u}}_m$ we will assume that $\tilde{\mathbf{v}} = \tilde{\mathbf{u}}_m \pm \boldsymbol{\varphi}$, where $\boldsymbol{\varphi} \in \mathbf{W}_0$ and $\boldsymbol{\varphi} = 0$ on Γ_6, and we will take the limit as $m \to \infty$. As a result we have

$$(\mathbf{T}(\tilde{\mathbf{u}}) - \mathbf{T}_*, \mathbf{D}\boldsymbol{\varphi}^T) = 0 \ \forall \mathbf{D}\dot{\boldsymbol{\varphi}} \in \vec{\mathbf{L}}_p$$

i.e., $\mathbf{T}_* = \mathbf{T}(\tilde{\mathbf{u}})$, and the operator $\mathbf{T}$ is weakly continuous. Since $t_N^{(k+1)}$ is determined from inequality (4.8) in terms of $\mathbf{T}^{(k+1)}$, the operator Q is weakly continuous.

Putting $\tilde{\mathbf{v}} = \tilde{\mathbf{u}}_{(k+1)} \pm \boldsymbol{\varphi}$ in Eq. (4.6), where $\boldsymbol{\varphi} \in \mathbf{W}_0$ and $\varphi_N = 0$ on Γ, we can to derive the inequality

$$\left|(\mathbf{T}^{(k+1)}, \mathbf{D}\boldsymbol{\varphi}^T)\right| \leq -\langle t_{\mathbf{f}}^{(k)}, |\varphi_{\mathbf{f}}|\rangle + |L(\boldsymbol{\varphi})|$$

Hence

$$\left\|\mathbf{T}^{(k+1)}; \vec{\mathbf{L}}_p\right\| \leq a\|L\| + b\left\|t_{\mathbf{f}}^{(k)}\right\|, \quad \|L\| = \|\rho\mathbf{S}\| + \|\mathbf{P}_n\|$$

Here, $||\rho\mathbf{S}||$, $||\mathbf{P}_n||$ and $||t_{\boldsymbol{f}}^{(k)}||$ are norms in the spaces $\mathbf{W}'$, $\mathbf{V}'_1$ and $\mathbf{V}'_6$. Since $||\mathbf{Dv};\ \vec{\mathbf{L}}_p||$ is the norm in $\mathbf{W}_0$, from equality (4.8) we have

$$\left\|t_N^{(k+1)}\right\| \leq c\left\|\mathbf{T}^{(k+1)}\right\| + d\|L\|$$

so that, taking the preceding inequality into account, we obtain

$$\left\|t_{\mathbf{f}}^{(k+1)}\right\| = \left\|f^{(k+1)} t_N^{(k+1)}\right\| \leq c_1\|f\|\left\|t_{\mathbf{f}}^{(k)}\right\| + c_2$$

$$c_1 = bc\|f\|, \quad c_2 = (ac+d)\|L\|, \quad \|f\| = \left\|f;\ L_\infty(\Gamma_6)\right\|$$

When $c_1||f|| < 1$ a constant $r > 0$ exists such that the mapping of Q transfers to itself the set $B_r \cap C*^-$, where B_r is a sphere of radius r, and C^{*-} is the cone of non-positive distributions in space V'_6. The closed sphere B_r is convex and compact in weak topology, since the space V'_6 is reflexive.

We will use the following theorem of Tikhonov.[6,23]

Theorem 6.1. *Suppose $\tilde{Q} : \Phi \subset U \to \Phi$ is a continuous mapping, where Φ is a non-empty, compact, convex set in locally convex space U. Then, the mapping of $\tilde{Q}$ has a fixed point in the set Φ.*

As a result, the operator Q has at least one fixed point in space V'_6. Taking into account the proved equivalence of the auxiliary problems, in particular Problems PW^t and PW, we can consider the existence of a weak solution of the quasi-stationary problem to be proved.

Theorem 6.2. *Suppose the conditions of Theorem 5.2 are satisfied. Then, a constant $f_* > 0$ is found such that, when $||f|| < f_*$, the quasi-static problem has at least one weak solution.*

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